COMMENTS ON "ABSOLUTE ANABELIAN CUSPIDALIZATIONS OF PROPER HYPERBOLIC CURVES"

Shinichi Mochizuki

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(1.) The argument given in Remark 1.18.2 [i.e., Remark 10 in the LaTeX Version], (iii), is somewhat sketchy and a bit misleading. This argument should read as follows:

(iii) Nevertheless, as was pointed out to the author by *A. Tamagawa*, even if X, Y are *not necessarily* Σ*-separated*, it is still possible to conclude, essentially from the arguments of [Tama], Corollary 2.10, Proposition 3.8, that:

Any *Frobenius-preserving* isomorphism α is *quasi-point-theoretic*.

Indeed, it suffices to give a *"group-theoretic" characterization* of the quasi-sections $D \subseteq \Pi_X$ which are decomposition groups of points $\in X^{\mathrm{cl}}$. Suppose, for simplicity, that X is *proper*. Write

$$\widetilde{X} \to X$$

for the *pro-finite étale covering* corresponding to Π_X. If $E \subseteq \Pi_X$ is a closed subgroup whose image in G_{k_X} is *open*, then let us write k_E for the finite extension field of k_X determined by this image. If $J \subseteq \Pi_X$ is an open subgroup, then let us write $X_J \to X$ for the covering determined by J and $J_\Delta \stackrel{\mathrm{def}}{=} J \bigcap \Delta_X$. If $J \subseteq \Pi_X$ is an open subgroup such that J_Δ is a characteristic subgroup of Δ_X, then we shall say that J is *geometrically characteristic*. Now let $J \subseteq \Pi_X$ be a *geometrically characteristic open subgroup*. Let us refer to as a *descent-group for* J any open subgroup $H \subseteq \Pi_X$ such that $J \subseteq H$, $J_\Delta = H_\Delta$. Thus, a descent-group H for J may be thought of as an intermediate covering $X_J \to X_H \to X$ such that $X_H \times_{k_H} k_J \cong X_J$. Write

$$X_J(k_J)^{\text{fld-def}} \subseteq X_J(k_J)$$

for the subset of k_J-valued points of X_J that do *not* arise from points $\in X_H(k_H)$ for any descent-group $H \neq J$ for J — i.e., the k_J-valued points whose *field of definition* is k_J with respect to *all possible "descended forms" of* X_J. Thus, if $\widetilde{x}$ is a closed point of $\widetilde{X}$ that maps to $x \in X_J(k_J)$, and we write $D_{\widetilde{x}} \subseteq \Pi_X$ for the *stabilizer in* Π_X [i.e., "decomposition group"] of $\widetilde{x}$, then it is a *tautology* that x maps to a point $\in X_{H_x}(k_{H_x})$ for $H_x \stackrel{\mathrm{def}}{=} D_{\widetilde{x}} \cdot J_\Delta$ $(\supseteq J)$ [so H_x forms a *descent-group for* J]; in particular, it follows immediately that:

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$$x \in X_J(k_J)^{\text{fld-def}} \iff D_{\tilde{x}} \subseteq J \iff H_x = J.$$

Now it follows immediately from this characterization of "fld-def" that if $J_1 \subseteq J_2 \subseteq \Pi_X$ are geometrically characteristic open subgroups such that $k_{J_1} = k_{J_2}$, then the natural map $X_{J_1}(k_{J_1}) \to X_{J_2}(k_{J_2})$ induces a map $X_{J_1}(k_{J_1})^{\text{fld-def}} \to X_{J_2}(k_{J_2})^{\text{fld-def}}$. Moreover, these considerations allow one to conclude [cf. the theory of [Tama]] that:

> A quasi-section $D \subseteq \Pi_X$ is a *decomposition group* of a point $\in X^{\text{cl}}$ if and and only if, for every geometrically characteristic open subgroup $J \subseteq \Pi_X$ such that $D \cdot J_\Delta = J$, it holds that $X_J(k_J)^{\text{fld-def}} \neq \emptyset$.

Thus, to render this characterization of decomposition groups *"group-theoretic"*, it suffices to give a "group-theoretic" criterion for the condition that $X_J(k_J)^{\text{fld-def}} \neq \emptyset$. In [Tama], the *Lefschetz trace formula* is applied to compute the cardinality of $X_J(k_J)$. On the other hand, if we use the notation "$|-|$" to denote the cardinality of a finite set, then one verifies immediately that

$$|X_J(k_J)| = \sum_H |X_H(k_H)^{\text{fld-def}}|$$

— where $H \supseteq J$ ranges over the *descent-groups* for J. In particular, by applying *induction* on $[\Pi_X : J]$, one concludes immediately from the above formula that $|X_J(k_J)^{\text{fld-def}}|$ may be computed from the $|X_H(k_H)|$, as H ranges over the *descent-groups* for J [while $|X_H(k_H)|$ may be computed, as in [Tama], from the *Lefschetz trace formula*]. This yields the desired "group-theoretic" characterization of the decomposition groups of Π_X.

(2.) In the proof of Theorem 1.16, (iii) [i.e., Theorem 1.1, (iii), in the LaTeX version], the phrase "$Z'_X \to X$, $Z'_Y \to Y$ are *diagonal coverings*" that appears at the beginning of this proof should read "$Z'_X \to X \times X$, $Z'_Y \to Y \times Y$ are *diagonal coverings*".

(3.) In the proof of Proposition 2.2, (i), it is asserted that one has a "*natural isomorphism* $H^1(k, \Delta_X^{\text{ab}}) \xrightarrow{\sim} J(k)^\wedge$ $(\supseteq J(k))$". In fact, however, at least from an *a priori* point of view, one only has a *natural injection* $(J(k) \subseteq)$ $J(k)^\wedge \hookrightarrow H^1(k, \Delta_X^{\text{ab}})$. This does not have any effect on the proof, however, since the proof only requires the use of this natural injection.

(4.) In the proof of Proposition 1.6, (i), the phrase "fact that $G_k^\dagger$ acts *faithfully* on M_X via the *cyclotomic character*" should read "fact that the image in $\text{Aut}(M_X)$ of the action of $G_k^\dagger$ on M_X via the *cyclotomic character* is *infinite*".